International Journal of Theoretical Physics, Vol. 32, No. 9, 1993

Algebraic Spinors and Directed Random Walks in the McKane–Parisi–Sourlas Theorem

Suemi Rodríguez-Romo[1,2]

Received August 12, 1992

We present the Dirac propagator as a random walk on an S^{D-1} sphere for Majorana spinors, even spinor space, Dirac spinors, and Chevalley–Crumeyrolle spinors built from Minkowski space. We propose the Dirac propagator constructed from Chevalley–Crumeyrolle spinors as the generators of a Markov process such that McKane–Parisi–Sourlas theorem can be applied to calculate the expectation values for functions of local times.

1. INTRODUCTION

The Dirac propagator can be represented as a continuum limit of a discrete directed random walk in which the directions of consecutive steps are correlated by appropriate rotation matrices (Jacobson, 1984, 1985; Ambjørn *et al.*, 1990). In this paper we generalize this concept using algebraic spinors defined on Minkowski space-time.

We describe Dirac propagators in terms of random variations of path direction and position, simultaneously. Thus, we deal with a random walk on a sphere of tangent vectors.

The equivalence between this random walk and the corresponding path integral formula has been well established in the Euclidean case (Jaroszewicz and Kurzepa, 1991).

In the case of Chevalley–Crumeyrolle algebraic spinors, the introduction of a new isotropic basis allows us to use them as the inverse of the generator of a Markov process. Therefore, the McKane–Parisi–Sourlas theorem can be used. In our approach even the Green function is valued in Grassmann algebra.

[1]Institut de Physique Théorique (Unité FYMA), Université Catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium.
[2]Permanent address: Facultad de Estudios Superiores Cuautitlán, Universidad Nacional Autónoma de México, Cuautitlán Izcalli, Estado de México, 54700 México.

902/32/9-1

2. DIRECTED RANDOM WALK REPRESENTATION

Let us write the Dirac propagator (in momentum space) in D-dimensional space for a fixed length of the path L as follow (Jaroszewicz and Kurzepa, 1991):

$$\frac{1}{m - i\gamma \cdot p} = \int_0^\infty e^{-m(L/D)} e^{i\gamma \cdot p(L/D)} \frac{dL}{D} \tag{1}$$

for a fermion of mass m and momentum p. Here $\{\gamma_\mu\}$, $\mu = 0, \ldots, D-1$, is the set of generators of a 2^D-dimensional Clifford algebra.

On the other hand we know that

$$e^{i\gamma \cdot p(L/D)} = \lim_{N \to \infty} \left(1 + \frac{iL}{ND} \gamma \cdot p\right)^N \tag{2}$$

and we can write (Jaroszewicz and Kurzepa, 1991)

$$\left(1 + \frac{iL}{ND} \gamma \cdot p\right) = \int dn \frac{1}{2} (1 + \gamma \cdot n) \left(1 + i \frac{L}{N} p \cdot n\right) \tag{3}$$

with $n \in S^{D-1}$ and the invariant measure on the sphere S^{D-1} is normalized such that

$$\int dn = 2 \tag{4}$$

We have figured out the propagator of a Dirac particle as described in terms of the path length L and a local propagation direction, i.e., a unit vector tangent to the path,

$$n(l) = \frac{dx(l)}{dl} \tag{5}$$

where $l \in [0, L]$ parametrizes the path.

In addition, by construction of the 2^D-dimensional Clifford algebra generated by $\{\gamma_\mu\}$, the so-called projection operator can be generalized to the primitive idempotent of the algebra.

The left minimal ideals defined by the primitive idempotent of the Clifford algebra are the algebraic spinors $\mathscr{S}(p, q)$. We can locally choose the algebraic spinorial basis of the D-dimensional left minimal ideal, so from (3) and (2) we get

$$e^{i\gamma \cdot p(L/D)} = \lim_{N \to \infty} \int \prod_{k=1}^{N} dn_k\, f(n_k) \left(1 + i \frac{L}{N} p \cdot n\right)^N \tag{6}$$

where we have discretized the unit vectors tangent to the path and $f(n_\alpha)$ is the primitive idempotent constructed on the fiber over $n_\alpha \in S^{D-1}$. In the limit $N \to \infty$ and from (6) it follows that

$$\int \frac{d^D p}{(2\pi)^D} e^{-ip\cdot(X'-X)} e^{i\gamma\cdot p(L/D)}$$
$$= \lim_{N\to\infty} \int \frac{d^D p}{(2\pi)^D} e^{-ip\cdot(X'-X)} \int \prod_{k=1}^{N} dn_k \, e^{i(L/N)p\cdot n_k} f(n_N)\cdots f(n_1) \qquad (7)$$

For Minkowski space-time we present the following particular cases for the R.H.S. of equation (7):

a. Majorana Spinors. Here we have

$$\lim_{N\to\infty} \int \frac{d^D p}{(2\pi)^D} e^{-ip\cdot(X'-X)} \int \prod_{k=1}^{N} dn_k \, e^{i(L/N)p\cdot n_k}$$
$$\times \frac{1}{2^N} [1 + \{e_3 e_4(1+e_1) + e_1\} n_N]$$
$$\times \cdots \times [1 + \{e_3 e_4(1+e_1) + e_1\} n_1] \qquad (8)$$

where hereafter $\{e_1, e_2, e_3, e_4\}$ is the orthogonal basis for the Minkowski space-time, such that $e_1^2 = e_2^2 = e_3^2 = -e_4^2 = 1$. Equation (8) is constructed as an operator over $\mathscr{C}(3, 1)$, the Clifford algebra generated by Minkowski space-time. In this case the idempotent of the algebra has been chosen as (Bugajska, 1986)

$$f_M \equiv \{\tfrac{1}{2}(1+e_1)\,\tfrac{1}{2}(1+e_3 e_4)\} \qquad (9)$$

b. Even Spinor Space. Here

$$\lim_{N\to\infty} \int \frac{d^D p}{(2\pi)^D} e^{-ip\cdot(X'-X)} \int \prod_{k=1}^{N} dn_k \, e^{i(L/N)p\cdot n_k}$$
$$\times \frac{1}{2^N} (1 + e_3 e_4 n_N) \times \cdots \times (1 + e_3 e_4 n_1) \qquad (10)$$

is constructed as an operator over $\mathscr{C}^E(3, 1)$, i.e., the even subalgebra of $\mathscr{C}(3, 1)$. In this case the idempotent of the algebra has been chosen as (Bugajska, 1986)

$$f_E = \tfrac{1}{2}(1 + e_3 e_4) \qquad (11)$$

c. Dirac Spinors. Here

$$\lim_{N\to\infty}\int\frac{d^Dp}{(2\pi)^D}e^{-ip\cdot(X'-X)}\int\prod_{k=1}^{N}dn_k\,e^{i(L/N)p\cdot n_k}$$
$$\times\frac{1}{2^N}\left[1+\{e_3e_4(1+e_1)+e_1\}n_N\right]$$
$$\times\cdots\times\left[1+\{e_3e_4(1+e_1)+e_2\}n_1\right] \tag{12}$$

is constructed as an operator over the $\mathscr{C}^c(3,1)$ algebra, which means that we can consider the Dirac spinors as elements of the left minimal ideal of $\mathscr{C}(4,1)$, because the complexified $\mathscr{C}(3,1)$ algebra $[\mathscr{C}^c(3,1)]$ is isomorphic to $\mathscr{C}(4,1)$. In other words, Minkowski space-time can be considered as a subspace of the space spanned by $\{e_0, e_1, e_2, e_3, e_4\}$ and we can use the same idempotent of the algebra as in the case of Majorana spinors.

d. Chevalley–Crumeyrolle Spinor Space. Here

$$\lim_{N\to\infty}\int\frac{d^Dp}{(2\pi)^D}e^{-ip\cdot(X'-X)}\int\prod_{k=1}^{N}dn_k\,e^{i(L/N)p\cdot n_k}$$
$$\times\frac{1}{2N}\left[(e_1-ie_2)(e_3-e_4)n_N\right]\times\cdots\times\left[(e_1-ie_2)(e_3-e_4)n_1\right] \tag{13}$$

is constructed as an operator over the $\mathscr{C}^c(3,1)$ algebra generated by the $\{e_1, e_2, e_3, e_4\}$ basis. In this case the idempotent of the algebra used is

$$f_{cc}=\tfrac{1}{4}(e_1-ie_2)(e_3-e_4) \tag{14}$$

The scalar products on the spinor space $\mathscr{S}(p,q)$ are defined as the maps $(\cdot,\cdot)_\pm : \mathscr{S}(p,q)\times\mathscr{S}(p,q)\to F$, where $F=f\mathscr{C}(3,1)f$.

The identity and the reflection on the Minkowski space-time induce antinvolutions of the associated Clifford algebra $\mathscr{C}(3,1)$. They are usually denoted by β_+ and β_-, respectively.

We define the scalar products $(\cdot,\cdot)_\pm$ according to the following formula:

$$(\psi,\varphi)_\pm=\omega_\pm\beta_\pm(\psi)\varphi, \qquad \forall\psi,\varphi\in\mathscr{S}(p,q) \tag{15}$$

where $\omega\in\mathscr{C}(3,1)$, such that $\omega_\pm\beta_\pm(f)\omega_\pm^{-1}=f$.

Thus, we can rewrite the R.H.S. of (7) as

$$\lim_{N\to\infty} \omega_{\pm}^{N} \int \frac{d^D p}{(2\pi)^D} e^{-ip\cdot(X'-X)} \int \prod_{k=1}^{N} dn_k\, e^{i(L/N)p\cdot n_k}$$

$$\times \omega_{\pm}^{N} \prod_{i=N}^{1} \beta_{\pm}(\zeta(n_i))\,\zeta(n_i), \qquad \forall \zeta(n_i)\in \mathscr{S}(p,q) \tag{16}$$

The transition function is defined as follows:

$$P(n_N, n_{N-1}) = \zeta(n_N)\,\beta_{\pm}(\zeta(n_{N-1})) \tag{17}$$

From the properties of $\mathscr{C}(3,1)$, P has the composition property

$$\int dn\, P(n_2, n)\, P(n, n_1) = P(n_2, n_1) \tag{18}$$

We interpret (17) as the probability of going from the direction given by n_{N-1} to the one given by n_N. So, (16) is equivalent to

$$\lim_{N\to\infty} \omega_{\pm}^{N} \int \frac{d^D p}{(2\pi)^D} e^{-ip\cdot(X'-X)} \int \prod_{k=1}^{N} dn_k\, e^{i(L/N)p\cdot n_k}$$

$$\times \beta_{\pm}(\zeta(n_N))\, P(n_N, n_{N-1}) \cdots P(n_2, n_1)\,\zeta(n_1) \tag{19}$$

Carrying out the integration over p, we get

$$\lim_{N\to\infty} \omega_{\pm}^{N} \int \prod_{k=1}^{N} dn_k\, \beta_{\pm}(\zeta(n_N))\, P(n_N, n_{N-1}) \cdots P(n_2, n_1)\,\zeta(n_1)$$

$$\times \delta^D\left(X' - X - L/N \sum_{k=1}^{N} n_k\right) \tag{20}$$

If we apply the operators $\zeta(n_N)$ and $\beta(\zeta(n_1))$ to the right and left sides of (20), respectively, we get the propagator of a Dirac particle subject to the constraint $f\zeta(n_i)=0$. Namely

$$G(L; X', n'; X, n)$$

$$= \lim_{N\to\infty} \omega_{\pm}^{N} \int \prod_{k=1}^{N} dn_k\, P(n', n_N) \cdots P(n_1, n)\, \delta^D\left(X' - X - \frac{L}{N}\sum_{k=1}^{N} n_k\right) \tag{21}$$

which describes propagation of a fermion starting at the point X in the direction n and arriving at X' in the direction n' after moving in a path of length L.

Let us stress that the differences between consecutive values of n_i are in no sense small and (21) can be understood as the convolution (over position X_i and directions n_i) of N propagators, for path lengths L/N for each of them.

3. MCKANE, PARISI, SOURLAS THEOREM

We think that the Dirac propagator, as a directed random walk constructed in the exterior or Grassmann algebra defined by the space of Chevalley–Crumeyrolle spinors, can be used to define the generator of a Markov process. Moreover, we can use the McKane–Parisi–Sourlas theorem (McKane, 1980; Parisi and Sourlas, 1979, 1980) to calculate expectation values of functions of local times in a random walk as a Gaussian integral of Grassmann valued fields where even the Green function is a Grassmann field.

Let $G^{-1}(L; X', n'; X, n)$ be the generator of a Markov process; then $\exp[-tG^{-1}(L; X', n'; X, n)]$ is the semigroup of transition probabilities for the process, so the McKane–Parisi–Sourlas theorem reads

$$\int_0^\infty dt\, E(F(\tau')1_{\omega(t)=j} | \, \omega(0)=i) = \int d\mu_G(\Phi)\, F(\Phi^2) \begin{bmatrix} \bar{\psi}_i \psi_j \\ \text{or} \\ \bar{\varphi}_i \varphi_j \end{bmatrix}$$

where $G = G(L; X', n'; X, n)$, t is time, i and j are the initial and final steps of the walk, and τ' is the set of local times spent in each step of the walk. Moreover, Φ is a field defined in each step of the walk like $\Phi^2 = (\phi_1^2, \ldots, \phi_n^2)$ and $\Phi_i^2 = \varphi_i \bar{\varphi}_i + \psi_i \bar{\psi}_i$, where φ_i and $\bar{\varphi}_i$ are complex-valued functions and ψ_i and $\bar{\psi}_i$ are Grassmann ones.

Finally, $F(\tau')$ is defined such that

$$|F(\tau')| \leqslant \text{const} \cdot \exp\left(-b \sum t_i\right) \qquad \text{for some} \quad b > 0$$

ACKNOWLEDGMENT

The technical assistance of C. Brichard is greatly appreciated.

REFERENCES

Ambjørn, J., Durhuus, B., and Jonsson, T. (1990). *Nuclear Physics B*, **330**, 509.
Bugajska, K. (1986). *Journal of Mathematical Physics*, **27**, 143.
Jacobson, T. (1984). *Journal of Physics A*, **17**, 2433.
Jacobson, T. (1985). *Journal of Mathematical Physics*, **26**, 1600.
Jaroszewicz, T., and Kurzepa, P. S. (1991). *Annals of Physics*, **210**, 255.
McKane, A. J. (1980). *Physics Letters*, **76A**, 22–24.
Parisi, G., and Sourlas, N. (1979). *Physics Letters*, **43**, 744–745.
Parisi, G., and Sourlas, N. (1980). *Journal de Physique Lettres* (Paris), **41**, L403–L406.